Wonderful Consequences of the Kerr Theorem

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Kerr's multi-particle solution is obtained on the base of the Kerr theorem. Choosing generating function of the Kerr theorem F as a product of partial functions F_i for spinning particles i=1,...k, we obtain a multi-sheeted, multi-twistorial space-time over M^4 possessing unusual properties. Twistorial structures of the i-th and j-th particles do not feel each other, forming a type of its internal space. Gravitation and electromagnetic interaction of the particles occurs via a singular twistor line which is common for twistorial structures of interacting particles. The obtained multi-particle Kerr-Newman solution turns out to be 'dressed' by singular twistor lines linked to surrounding particles. We conjecture that this structure of space-time has the relation to a stringy structure of vacuum and opens a geometrical way to quantum gravity.

Introduction. It has been mentioned long ago that the Kerr-Newman solution displays some relationships to the quantum world. It is the anomalous gyromagnetic ratio $g = 2$, as that of the Dirac electron [1], stringy structures [2,3] and other features allowing one to construct a semiclassical model of the extended electron [3–6,8,9] which has the Compton size and possesses the wave properties [3,5,10].

One of the mysteries of the Kerr geometry is the existence of two sheets of space-time, $(+)$ and $(−)$, on which the dissimilar gravitation (and electromagnetic) fields are realized, and fields living on the $(+)$-sheet do not feel the fields of the $(−)$-sheet. Origin of this twofoldedness lies in the Kerr theorem, generating function F of which for the Kerr-Newman solution has two roots which determine two different twistorial structures on *the same* space-time.

In this letter we describe the Kerr's multi-particle solution. Choosing generating function F of the Kerr theorem as a product of partial functions F_i for spinning particles i=1,...k, we obtain multi-sheeted, multi-twistorial space-time over M^4 possessing unusual properties. Twistorial structures of the i-th and j-th particles do not feel each other, forming a type of its internal space. Gravitation and electromagnetic interaction of the particles occurs via a singular twistor line.

This unusual structure of space-time is the direct generalization of Kerr's twofoldedness and we conjecture that it displays also a relation to quantum physics.

The Kerr-Newman metric can be represented in the Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h k_\mu k_\nu, \qquad (1)$$

where $\eta_{\mu\nu}$ is metric of auxiliary Minkowski space-time M^4,

$$h = (mr - e^2/2)/(r^2 + a^2 \cos^2 \theta), \qquad (2)$$

and $k_\mu(x)$ is a twisting null field, which is tangent to the Kerr principal null congruence (PNC) which is geodesic and shear-free [11,13]. PNC is determined by the complex function $Y(x)$ via the one-form

$$e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv = Pk_\mu dx^\mu \qquad (3)$$

where u, v, $\bar{\zeta}$, ζ are the null Cartesian coordinates. Here P is a normalizing factor for k_μ which provide $k_0 = 1$ in the rest frame.[1] The null rays of the Kerr congruence are twistors. Form of the Kerr PNC is shown on Fig. 1.

The Kerr theorem [11,12] allows one to describe the Kerr geometry in twistor terms [12,13].
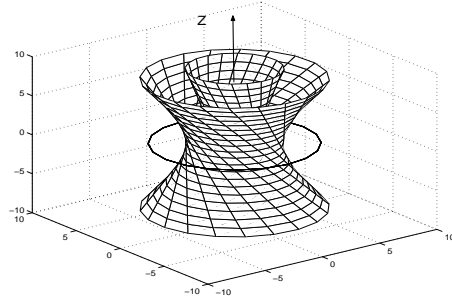

FIG. 1. The Kerr singular ring and 3-D section of the Kerr principal null congruence. Singular ring is a branch line of space, and PNC propagates from "negative" sheet of the Kerr space to "positive " one, covering the space-time twice.

It claims that any geodesic and shear-free null congruence in Minkowski space-time is defined by a function $Y(x)$ which is a solution of the equation

$$F = 0, \qquad (4)$$

where $F(Y, \lambda_1, \lambda_2)$ is an arbitrary holomorphic function of the projective twistor coordinates

$$Y, \quad \lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}. \qquad (5)$$

In the Kerr-Schild backgrounds the Kerr theorem acquires a broader content [13,7,11], allowing one to determine the normalizing function P and complex radial distance $\tilde{r} = r + ia \cos \theta$,

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[1] We replace the factors P from e^3 to function h, so h in [11] differs by factor P^{-2}.

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$$P = \partial_{\lambda_1} F - \bar{Y}\partial_{\lambda_2} F, \quad \tilde{r} = PZ^{-1} = - \quad dF/dY \quad (6)$$

which are important characteristics of the corresponding solutions. The position of singular lines, caustics of PNC, corresponds to $\tilde{r} = 0$, and is determined by the system of equations $F = 0;\quad dF/dY = 0$.

The proof of the Kerr theorem in the extended version adapted to the Kerr-Schild formalism is given in [13].

In the original paper [11], the following generating function F was considered

$$F \equiv \phi(Y) + (qY + c)(d\zeta - Ydv) - (pY + \bar{q})(u + Y\bar{\zeta}). \quad (7)$$

The parameters $p, q, \bar{q}, c$ are related to the Killing vector of the solution $K^\mu \partial_\mu = c\partial_u + \bar{q}\partial_\zeta + q\partial_{\bar{\zeta}} - p\partial_v$, and $P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c$. For the stationary Kerr solution $p = c = 2^{-1/2},\quad q = \bar{q} = 0$ and $\tilde{r} = r + ia\cos\theta$.

It was shown in [7,14] that function $\phi(Y)$ in (7) has to be at most quadratic in Y to provide singular lines to be confined in a restricted region, which corresponds to the Kerr PNC up to the Lorentz boosts, orientations of angular momenta and the shifts of origin.

In the papers [7,13] another form for this function was suggested $F = (\lambda_1 - \lambda_1^0)\check{K}\lambda_2 - (\lambda_2 - \lambda_2^0)\check{K}\lambda_1$ which is related to the Newman-initiated [15] complex representation of the Kerr geometry. In this case, function $F(Y)$ can be expressed via the set of parameters q which determine the motion and orientation of the Kerr spinning particle and takes the form $F(Y|q) = A(x|q)Y^2 + B(x|q)Y + C(x|q)$. The equations (4) can be resolved explicitly, leading to two roots $Y = Y^\pm(x|q)$ which correspond to two sheets of the Kerr space-time. The root $Y^+(x)$ determines via (3) the out-going congruence on the $(+)$-sheet, while the root $Y^-(x)$ gives the in-going congruence on the $(-)$-sheet. Therefore, function F may be represented in the form $F(Y|q) = A(x|q)(Y - Y^+)(Y - Y^-)$, which allows one to obtain all the required functions of the Kerr solution in explicit form. The detailed form of $Y^\pm(x|q)$ is not important for our treatment here and may be found in [13].

Multi-twistorial space-time. Selecting an isolated i-th particle with parameters q_i, one can obtain the roots $Y_i^\pm(x)$ of the equation $F_i(Y|q_i) = 0$ and express F_i in the form

$$F_i(Y) = A_i(x)(Y - Y_i^+)(Y - Y_i^-). \quad (8)$$

Then, substituting the $(+)$ or $(-)$ roots $Y_i^\pm(x)$ in the relation (3), one determines congruence $k_\mu^{(i)}(x)$ and consequently, the Kerr-Schild ansatz (1) for metric

$$g_{\mu\nu}^{(i)} = \eta_{\mu\nu} + 2h^{(i)}k_\mu^{(i)}k_\nu^{(i)}, \quad (9)$$

and finally, the function $h^{(i)}(x)$ may be expressed in terms of $\tilde{r}_i = -d_Y F_i$, (6), as follows

$$h^{(i)} = \frac{m}{2}\left(\frac{1}{\tilde{r}_i} + \frac{1}{\tilde{r}_i^*}\right) + \frac{e^2}{2|\tilde{r}_i|^2}. \quad (10)$$

Electromagnetic field is given by the vector potential

$$A_\mu^{(i)} = \Re e(e/\tilde{r}_i)k_\mu^{(i)}. \quad (11)$$

What happens if we have a system of k particles? One can form the function F as a product of the known blocks $F_i(Y)$,

$$F(Y) \equiv \prod_{i=1}^{k} F_i(Y). \quad (12)$$

The solution of the equation $F = 0$ acquires $2k$ roots $Y_i^\pm$, and the twistorial space turns out to be multi-sheeted.
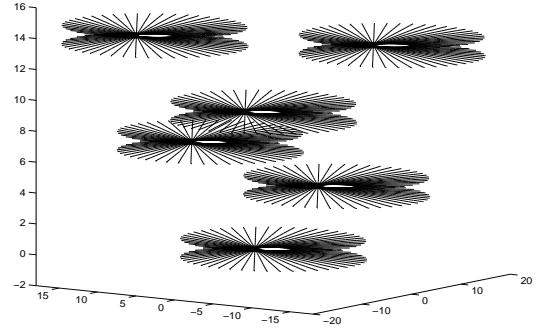

FIG. 2. Multi-sheeted twistor space over the auxiliary Minkowski space-time of the multi-particle Kerr-Schild solution. Each particle has twofold structure.

The twistorial structure on the i-th $(+)$ or $(-)$ sheet is determined by the equation $F_i = 0$ and does not depend on the other functions $F_j,\quad j \neq i$. Therefore, the particle i does not feel the twistorial structures of other particles. Similar, the condition for singular lines $F = 0$, $d_Y F = 0$ acquires the form

$$\prod_{l=1}^{k} F_l = 0, \qquad \sum_{i=1}^{k}\prod_{l\neq i}^{k} F_l d_Y F_i = 0 \quad (13)$$

and splits into k independent relations

$$F_i = 0, \quad \prod_{l\neq i}^{k} F_l d_Y F_i = 0. \quad (14)$$

One sees, that i-th particle does not feel also singular lines of other particles. The space-time splits on the independent twistorial sheets, and therefore, the twistorial structure related to the i-th particle plays the role of its "internal space".

It looks wonderful. However, it is a direct generalization of the well known twofoldedness of the Kerr space-time which remains one of the mysteries of the Kerr solution for the very long time.

For spinning particles $|a| >> m$ and the Kerr's black hole horizons disappear, there appears the old problem

of the source of Kerr solution with the alternative: either to remove this twofoldedness or to give it a physical interpretation. By truncation of the negative sheet, there appears the source in the form of relativistically rotating disk [4], bubble [8] or bag [16].

Alternative way is to retain the negative sheet, treating it as the sheet of advanced fields. In this case the source of spinning particle turns out to be the Kerr singular ring (circular string, [3,10]) with the electromagnetic excitations in the form of traveling waves which generate spin and mass of the particle (microgeon model [5,10]).

Multi-particle Kerr-Schild solution. Using the Kerr-Schild formalism with the considered above generating functions $\prod_{i=1}^{k} F_i(Y) = 0$, one can obtain the exact asymptotically flat multi-particle solutions of the Einstein-Maxwell field equations. Since congruences are independent on the different sheets, the congruence on the i-th sheet retains to be geodesic and shear-free, and one can use the standard Kerr-Schild algorithm of the paper [11]. One could expect that result for the i-th sheet will be in this case the same as the known solution for isolated particle. Unexpectedly, there appears a new feature having a very important consequence.

Formally, we have only to replace F_i by $F = \prod_{i=1}^{k} F_i(Y) = \mu_i F_i(Y)$, where $\mu_i = \prod_{j\neq i}^{k} F_j(Y)$ is a normalizing factor which takes into account the external particles. However, in accordance with (6) this factor μ_i will appear also in the function $\tilde{r} = -d_Y F = -\mu_i d_Y F_i$, and in the function $P = \mu_i P_i$.

So, we obtain the different result

$$h_i = \frac{m_i(Y)}{2\mu_i^3}\left(\frac{1}{\tilde{r}_i} + \frac{1}{\tilde{r}_i^*}\right) + \frac{(e/\mu_i)^2}{2|\tilde{r}_i|^2}, \qquad (15)$$

$$A_\mu^{(i)} = \Re e \frac{e}{\mu_i \tilde{r}_i} k_\mu^{(i)} \qquad (16)$$

which looks like a renormalization of the mass m and charge e.[2]

This fact turns out to be still more intriguing if we note that μ_i is not constant, but a function of Y_i. We can specify its form by using the known structure of blocks F_i

$$\mu_i(Y_i) = \prod_{j\neq i} A_j(x)(Y_i - Y_j^+)(Y_i - Y_j^-). \qquad (17)$$

The roots Y_i and $Y_j^{\pm}$ may coincide for some values of Y_i, which selects a common twistor for the sheets i and j. Assuming that we are on the i-th (+)-sheet, where congruence is out-going, this twistor line will also belong to

the in-going $(-)$-sheet of the particle j. The metric and electromagnetic field will be singular along this twistor line, because of the pole $\mu_i \sim A(x)(Y_i^+ - Y_j^-)$. Therefore, interaction occurs along a light-like Schild string which is common for twistorial structures of both particles. The field structure of this string is similar to the well known structure of pp-wave solutions.

The equations (1), (2) and (3) give the *exact multi-particle solution of the Einstein-Maxwell field equations.* It follows from the fact that the equations were fully integrated out in [11] and expressed via functions P and Z before (without) concretization of the form of congruence, under the only condition that it is geodesic and shear free. In the same time the Kerr theorem determines the functions P and Z via generating function F, eq.(6), and the condition of reality for metric may be provided by a special choice of the free function $m(Y)$.

The obtained multi-particle solutions show us that, in addition to the usual Kerr-Newman solution for an isolated spinning particle, there is a series of the exact 'dressed' Kerr-Newman solutions which take into account surrounding particles and differ by the appearance of singular twistor strings connecting the selected particle to external particles. This is a new gravitational phenomena which points out on a probable stringy (twistorial) texture of vacuum and may open a geometrical way to quantum gravity.
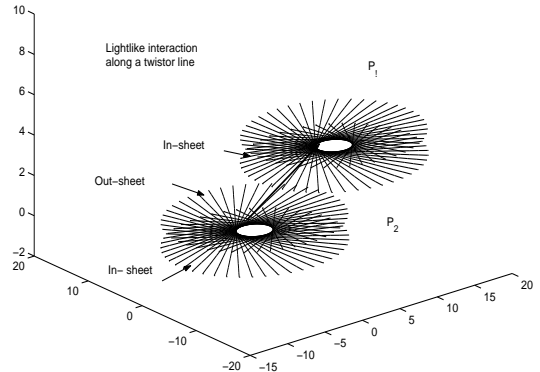

FIG. 3. Schematic representation of the lightlike interaction via a common twistor line connecting out-sheet of one particle to in-sheet of another.

The number of surrounding particles and number of blocks in the generating function F may be assumed countable. In this case the multi-sheeted twistorial spacetime will possess the properties of the multi-particle Fock space.

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[2]Function $m_i(Y)$ is free and satisfies the condition $(m_i)_{,\bar{Y}} = 0$. It and has to be chosen in the form $m_i(Y) = m_0\mu_i^3$ to provide reality of metric.

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